Mathl Modelling, Vol. 9, No. 11, pp. 799–811, 1987
Printed in Great Britain.

0270-0255/87 $3.00 + 0.00

NEW MODELS FOR SHALLOW FOUNDATIONS

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(*Received April 1986*)

Communicated by X. J. R. Avula

Abstract—Due to the recent development in construction techniques, fabrications of precast concrete units and numerical analysis, new foundations models were proposed. It is believed that these models will have higher bearing capacity and produce less settlement than similar conventional ones, resulting in an economical design. The present study presents the analysis of the stress distribution below different models. The results showed a reduction in the vertical, horizontal and shear stresses within a depth equal to the foundation width up to 20, 45 and 109%, respectively.

NOMENCLATURE

α = Angle in the x,z-plane
B = Half-width of the foundation
β = Angle in x,y-plane
C = Constant
d = Distance
$d\alpha$ = Increment
$d\beta$ = Increment
ds = Increment
M = Point where the stresses are analysed
N = Point where the load is applied
ν = Poisson's ratio
(O,x,z) = System of coordinates
(O,r,z) = System of coordinates
$\overline{ON}$ = Vector
P = Load applied on the foundation
q = Point load or line load
q_0 = Point load or line load
R = Radius
R_1,R_2 = Distances
S = Variable in the x-direction
w = Elongation
σ_x = Component of stress in the x-direction
σ_z = Component of stress in the z-direction
z = Depth
τ_{xz} = Shear stress in the x,z-plane

INTRODUCTION

The art of foundation engineering existed as early as when life was created on earth. At that time, and based on human feeling and judgement, flat foundations were extensively used to support walls. Due to the rapid increase in population in all parts of the world and pressing needs for constructing high-rise buildings in some crowded cities, other models of foundations should be considered if they prove to have better performance than the conventional flat ones.

As a result of the recently developed numerical techniques, computer facilities and highly sophisticated construction techniques, considering non-flat foundations is no longer a hypothetical problem. If such foundations proved to possess higher bearing capacity, and experience less settlements compared to the conventional flat ones, developing construction techniques should be the second goal.

The present paper presents an analysis of vertical, horizontal and shear stresses below the non-flat foundations, and some conclusions drawn.

ANALYSIS

In the present study, several models were considered (see Fig. 1). The soil was assumed to be homogeneous, isotropic and possess linear stress–strain behaviour. A two-dimensional analysis was conducted to simulate the case of strip footings.

In order to determine the stresses in the soil as a result of the loads supported by the above-mentioned models, the theory of elasticity has been used. The stress function was the solution to the equations of equilibrium with respect to the boundary conditions and the compatibility equation. The soil behaviour was simulated by Winkler's model.

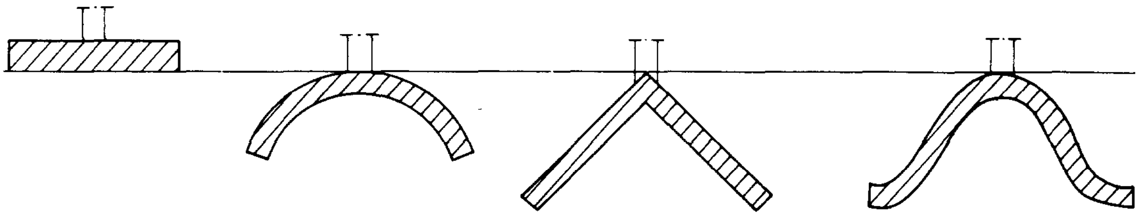

Fig. 1. Foundation models.

The classical problem of Boussinesq, dealing with a normal force applied at a plane boundary of a semi-infinite mass, was solved by superposing solutions, derived from Kelvin and Boussinesq. The Kelvin solution was used to determine stresses due to a force applied at a great distance from the ground surface, while the Boussinesq solution was used where the force acts at the gound surface. The solution proposed by Melan [1] and Mindlin [2] is applicable for the case where the load is near the ground surface.

Figure 2 shows a strip footing of width $2B$ and subjected to a vertical line load P. The stress on the ground surface is given by

$$q = \frac{P}{2B}, \tag{1}$$

while the vertical stress at point M at depth z (see Fig. 2) can be computed as follows, using the Boussinesq equation:

$$\sigma_z = \frac{2q}{\pi}\frac{z^3}{R_1^4}. \tag{2}$$

Due to an incremental part of the load of width ds and intensity q which acts at a distance s from the centre of the load (see Fig. 2), the vertical stress increment at point M is given by

$$d\sigma_z = \frac{2q}{\pi}\frac{z^3 ds}{R_1^4}. \tag{3}$$

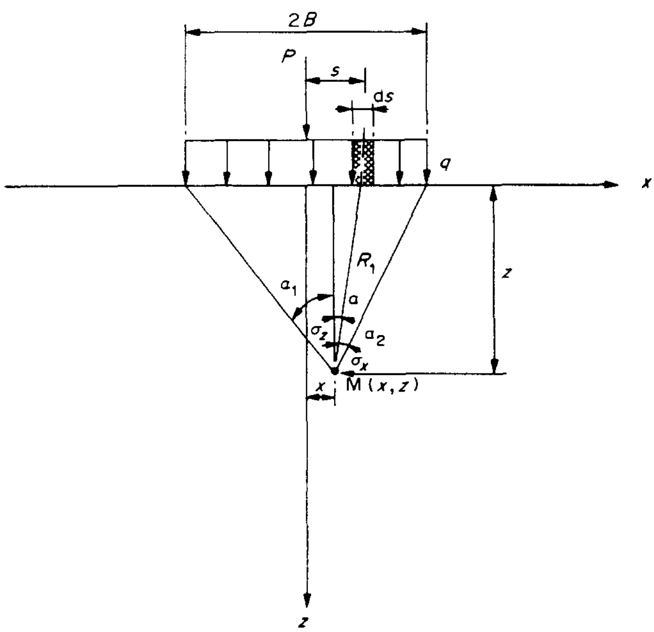

Fig. 2. Analysis of flat footing.

Using the polar coordinates and calculating $d\sigma_z$ as a function of $d\alpha$, the following equation can be deduced:

$$\alpha = \frac{s - x}{z}$$

or

$$(s - x) = z \tan \alpha, \tag{4}$$

$$\cos \alpha = \frac{z}{R_1}$$

or

$$R_1 = \frac{z}{\cos \alpha}, \tag{5}$$

$$\alpha_1 = \tan^{-1}\left(\frac{-x - B}{z}\right), \tag{6}$$

$$\alpha_2 = \tan^{-1}\left(\frac{-x + B}{z}\right), \tag{7}$$

where α_1 and α_2 are the lower and upper limits of the angle α. Differentiation of equation (4) yields

$$ds = \frac{z}{\cos^2 \alpha} d\alpha. \tag{8}$$

The substitution of equations (5) and (8) in equation (3), yields

$$d\sigma_z = \frac{2q}{\pi} \cos^2 \alpha \, d\alpha; \tag{9}$$

integrating equation (9) between the limits of α_1 and α_2, the vertical stress at point M can thus be obtained:

$$\sigma_z = \int_{\alpha_1}^{\alpha_2} \frac{2q}{\pi} \cos^2 \alpha d\alpha; \tag{10}$$

knowing

$$\int \cos^2 \alpha \, d\alpha = \frac{1}{2} + \frac{\sin 2\alpha}{4}, \tag{11}$$

equation (10) yields

$$\sigma_z = \frac{2q}{\pi}\left[\frac{\alpha_1 - \alpha_2}{z} + \frac{1}{4}(\sin 2\alpha_2 - \sin 2\alpha_1)\right]; \tag{12}$$

knowing

$$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2} \tag{13}$$

and substituting equations (1) and (13) in equation (12),

$$\sigma_z = \frac{P}{2\pi B}\left[(\alpha_2 - \alpha_1) + \sin\left(\frac{\alpha_2-\alpha_1}{2}\right)\cos\left(\frac{\alpha_2-\alpha_1}{2}\right)\right]. \tag{14}$$

In a similar manner the increment of the horizontal $d\sigma_x$ stress is given by

$$d\sigma_x = \frac{2q}{\pi}\frac{z(s-x)^2}{R^4}\,ds; \tag{15}$$

substituting equations (4), (5) and (8) in equation (15), then

$$d\sigma_x = \frac{2qz}{\pi}\frac{z^2\tan^2\alpha}{\frac{z^4}{\cos^4\alpha}}\frac{z}{\cos^2\alpha}\,d\alpha$$

or

$$d\sigma_x = \frac{2q}{\pi}(\sin^2\alpha\,d\alpha). \tag{16}$$

Integrating equation (16) between the limits of α_1 and α_2, the horizontal stress at point M can be obtained:

$$\sigma_x = \frac{2q}{\pi}\int_{\alpha_1}^{\alpha_2}\sin^2\alpha\,d\alpha. \tag{17}$$

In order to calculate the shear stress at point M, τ_{xz}, a similar analysis can be conducted:

$$\tau_{xz} = \frac{2q}{\pi}\frac{xz^2}{R^4}, \tag{18}$$

$$\begin{aligned} d\tau_{xz} &= \frac{2q}{\pi}\frac{z^2(s-x)}{R^4}\,ds \\ &= \left(\frac{2q}{\pi}z^2\right)\frac{a\tan\alpha}{\frac{z^4}{\cos^4\alpha}}\frac{z}{\cos^2\alpha}\,d\alpha \\ &= \frac{2q}{\pi}\sin\alpha\cos\alpha\,d\alpha; \end{aligned} \tag{19}$$

thus

$$\tau_{xz} = \frac{2q}{\pi}\int_{\alpha_1}^{\alpha}\sin\alpha\cos\alpha\,d\alpha \tag{20}$$

or

$$\tau_{xz} = \frac{q}{\pi}\{(\sin^2\alpha)\}_{\alpha_2}^{\alpha_1}$$

or

$$\tau_{xz} = \frac{P}{2\pi B}(\sin^2\alpha_2 - \sin^2\alpha_1). \tag{21}$$

The horizontal component stress σ_y is given by the following equation:

$$\sigma_y = \nu(\sigma_x + \sigma_z) = \nu\frac{q}{\pi B}(\alpha_2 - \alpha_1), \tag{22}$$

where ν = Poisson's ratio.

The above analysis can be extended to the case of a cylindrical strip footing, Fig. 3(a). It is assumed that the footing supports a line load P, per unit length of the footing. Due to symmetric loading and the shape of the footing, uniform settlement is expected (Winkler model).

Referring to Fig. 3(b), the footing will be divided into smaller segments, each having an angle $d\alpha$. Furthermore,

$2B$ = the width of the footing,

R = the radius of the circle,

s = the horizontal coordinate of a point N on the footing,

q_0 = the contact pressure at point N,

α = the central angle for the element N

and

$-\alpha_0, \alpha_0$ = the lower and upper limits of the angle α.

Thus,

$$s = R\sin\alpha \tag{23}$$

and

$$\alpha_0 = \sin^{-1}\left(\frac{B}{R}\right). \tag{24}$$

Differentiating equation (24), the following is obtained:

$$ds = R\cos\alpha\, d\alpha, \tag{25}$$

then

$$q = q_0\, ds = q_0 R\cos\alpha\, d\alpha. \tag{26}$$

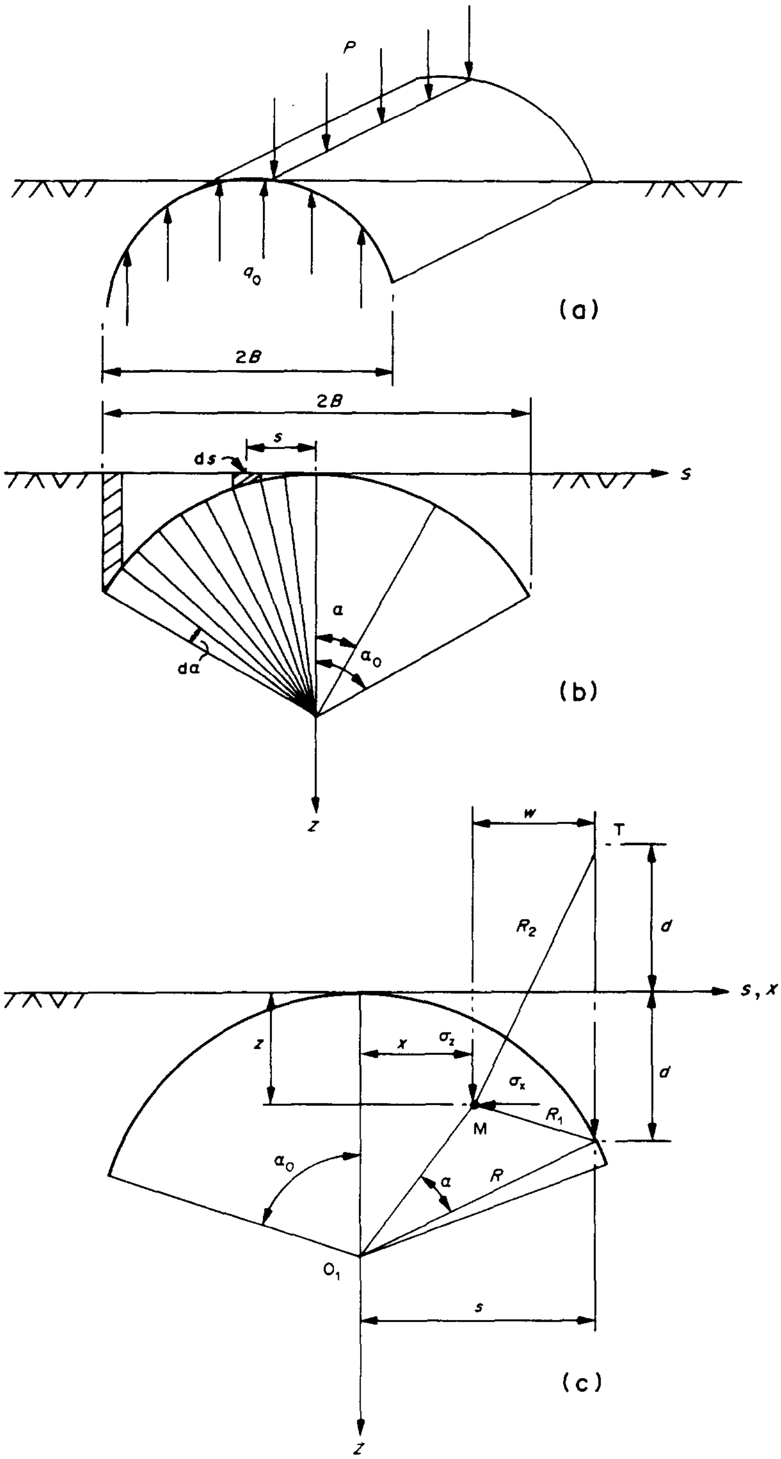

Fig. 3. (a) Cylindrical strip footing. (b) A sector divided into equal portions of dx. (c) Vectors and stresses.

For equilibrium of the cylindrical foundation, the total vertical load acting on the soil must be equal to the initial load per unit length, thus

$$P = \sum q\,\mathrm{d}s = \int_{-\alpha_0}^{\alpha_1} q_0 R\cos\alpha\,\mathrm{d}\alpha = \int_{-\alpha_0}^{\alpha_1} \frac{P}{2\beta} R\cos\alpha\,\mathrm{d}\alpha$$
$$= 2q_0 R\sin\alpha_0. \tag{27}$$

The stresses due to q at point M in Fig. 3(c), can be evaluated by using the equations developed by Melan [1], where $\sigma_z = V\frac{q}{\pi}$ by replacing x by $(s - x) = w$, thus

$$\sigma_z = \frac{1}{2(1-v)}\left[\frac{(z-d)^3}{R_1^4} + \frac{(z+d)(z+d)^2 + 2dz}{R_2^4} - \frac{8dz(d+z)w^2}{R_2^6}\right.$$
$$\left. = \frac{1-2v}{4(1-v)}\left(\frac{z-d}{R_1^2} + \frac{3z+d}{R_2^2} - \frac{4zw^2}{R_2^4}\right)\right]. \tag{28}$$

Referring to Fig. 3(c),

d = the vertical distance where the load acts,

$\overline{\mathrm{MN}} = R_1,$

$\overline{\mathrm{MT}} = R_2$

and

$$\overline{\mathrm{NT}} = 2d \tag{29}$$

in the (O_1, x, z) system, the vectors (O, N) *and* (O, M) *are equal to*

$$\overline{\mathrm{O_1N}} = (R\sin\alpha, R\cos\alpha)$$

and

$$\overline{\mathrm{O_1M}} = (x, r - z);$$

in the (O, x, z) system,

$$\overline{\mathrm{ON}} = (\mathrm{R}\sin\alpha, R - R\cos\alpha),$$
$$\overline{\mathrm{OM}} = (\mathrm{x}, \mathrm{z}) = \mathrm{const},$$
$$\overline{\mathrm{OT}} = (x_{\mathrm{N}}, -z_{\mathrm{N}}) = (R\sin\alpha, R\cos\alpha - R),$$
$$\overline{\mathrm{MN}} = \overline{\mathrm{ON}} - \overline{\mathrm{OM}} = ((R\sin\alpha - x), (R - R\cos\alpha - z))$$

and

$$\overline{\mathrm{MT}} = \overline{\mathrm{OT}} - \overline{\mathrm{OM}} = ((R\sin\alpha - x), (R\cos\alpha - R - z));$$

then the values of R_1, R_2, d and W are equal to

$$R_1 = \overline{\mathrm{MN}} = ((R\sin\alpha - x)^2 + (R - R\cos\alpha - z)^2)^{1/2}, \tag{30}$$

$$R_2 = \overline{\mathrm{MT}} = ((R\sin\alpha - z)^2 + (R\cos\alpha - R - z)^2)^{1/2}, \tag{31}$$

$$d = Z_{\mathrm{N}} = R - R\cos\alpha \tag{32}$$

and

$$W = R\sin - x. \tag{33}$$

As for a strip footing, for an increment $d\alpha$, the vertical stress increment is obtained using equations (27) and (28) as

$$d\sigma_z = A\frac{q}{\pi}ds = \frac{AP}{2B\pi}R\cos\alpha\, d\alpha; \tag{34}$$

therefore, the total vertical stress is equal to the integration of the above equation from $-\alpha_0$ to α_0, thus

$$\sigma_z = \frac{PR}{2B\pi}\int_{-\alpha_0}^{\alpha_0}\cos\alpha\left\{\frac{1}{2(1-\nu)}\left[\frac{(z+d)^3}{R_1^4} + \frac{(z+d)((z+d)^2+2dz)}{R_2^4} - \frac{8dz(d+2)w^2}{R_2^6}\right]\right.$$
$$\left. + \frac{1-2\nu}{4(1-\nu)}\left(\frac{z-d}{R_1^2} + \frac{3z+d}{R_2^4} - \frac{4zw^2}{R_2^6}\right)\right\}, \tag{35}$$

where R_1, R_2 and d are given by equations (30)–(32). Similarly, for the horizontal components,

$$\sigma_x = \frac{PR}{2B\pi}\int_{-\alpha_0}^{\alpha_0}\cos\alpha\left\{\frac{1}{2(1-\nu)}\left[\frac{(z-d)w^2}{R_1^4} + \frac{(z+d)(w^2+2d^2)-2dw^2}{R_2^4} + \frac{8dz(d+z)w^2}{R_2^6}\right]\right.$$
$$\left. + \frac{1-2\nu}{4(1-\nu)}\left(\frac{d-z}{R_1^2} + \frac{z+3d}{R_2^2} + \frac{4zw^2}{R_2^4}\right)\right\}; \tag{36}$$

and for the shear stress τ_{xz},

$$\tau_{xz} = \frac{PR}{2B\pi}\int_{-\alpha_0}^{\alpha_0}\cos\alpha\left\{\frac{1}{2(1-\nu)}\left[\frac{(z-d)^2}{R_1^4} + \frac{z^2-2dz-d^2}{R_2^4} + \frac{8dz(d+z)^2}{R_2^6}\right]\right.$$
$$\left. + \frac{1-2\nu}{4(1-\nu)}\left(\frac{1}{R_1^2} - \frac{1}{R_2^2} + \frac{4z(d+z)}{R_2^4}\right)\right\}. \tag{37}$$

The value of each integral was obtained using numerical analysis techniques.

A similar analysis was conducted on a triangular strip footing, as shown in Figs 4(a, b). Thus,

$$q = q_0\, ds. \tag{38}$$

To satisfy the equilibrium conditions, the sum of the vertical loads must be equal to zero, therefore

$$\sum q - P = 0, \tag{39}$$

where

$$q_0 = \frac{P}{2B}/\text{unit area} \tag{40}$$

and

$$q = \frac{P}{2B}ds/\text{unit length}. \tag{41}$$

In order to determine the stress components at point M, which is located below the ground level, a similar analysis of the cylindrical footing is performed.

Referring to Fig. 4(b),

$$R_1 = \overline{\text{MN}}, \quad R_2 = \overline{\text{TM}} \quad \text{and} \quad \overline{\text{TN}} = 2d, \tag{42}$$

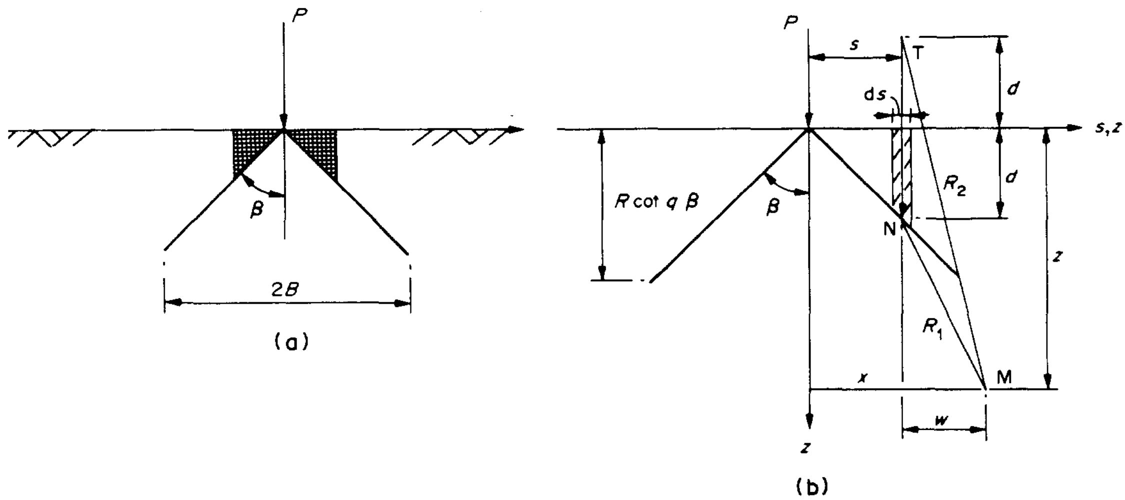

Fig. 4. (a) Triangular strip footing. (b) Vectors and stresses.

where

$$d = \frac{s}{\tan\beta}, \tag{43}$$

d is always positive; if s is positive, equation (43) can be rewritten as

$$d = \frac{s}{\tan\beta} \tag{44}$$

and if s is negative,

$$d = \frac{-s}{\tan\beta}. \tag{45}$$

The following vectors can be written:

$$\overline{\mathrm{OM}} = (x, z), \tag{46}$$

$$\overline{\mathrm{ON}} = (s, d) \tag{47}$$

$$\overline{\mathrm{OT}} = (s, -d), \tag{48}$$

$$\overline{\mathrm{MN}} = \overline{\mathrm{ON}} - \overline{\mathrm{OM}} = (s - x, z - d) \tag{49}$$

and

$$\overline{\mathrm{TN}} = \overline{\mathrm{ON}} - \overline{\mathrm{OT}} = (s - x, z + d). \tag{50}$$

Assume that

$$W = s - x, \tag{51}$$

then the moduli of the vectors $\overline{\mathrm{NM}}$ and $\overline{\mathrm{TM}}$ are equal to

$$R_1 = \overline{\mathrm{NM}} = [(s - x)^2 + (z - d)^2]^{1/2} \tag{52}$$

and

$$R_2 = \overline{\mathrm{TN}} = [(x - s)^2 + (2 + d)^2]^{1/2}, \tag{53}$$

assuming

$$C_2 = \frac{P}{2\pi B}. \tag{54}$$

In order to determine the vertical stress increase σ_2 at point M, integrating $d\sigma_z$ from $-B$ to $+B$, yields

$$\sigma_z = \int_{-B}^{B} C_2 \left\{ \frac{1}{2(1-\nu)} \left[\frac{(z-d)^3}{R_1^4} + \frac{(z+d)(z+d)^2 + 2dz}{R_2^4} - \frac{8dz(d+z)w^2}{R_2^6} \right] + \frac{1-2\nu}{4(1-\nu)} \left(\frac{z-d}{R_1^2} + \frac{3z+d}{R_2^2} - \frac{4zw^2}{R_2^4} \right) \right\} ds. \tag{55}$$

Similarly, for the horizontal component of the stress,

$$\sigma_x = \int C_2 \left\{ \frac{1}{1-2\nu} \left[\frac{(z-d)w^2}{R_1^4} + \frac{(z+d)(w^2+2d^2) - 2dz^2}{R_2^4} - \frac{8dz(d+z)}{R_2^6} \right] + \frac{1-2\nu}{4(1-\nu)} \left(\frac{d-z}{R_1^2} + \frac{z+3d}{R_2^2} + \frac{4zw^2}{R_2^4} \right) \right\} ds \tag{56}$$

and for the shear stress,

$$\tau_{xz} = \int_{-B}^{B} C_2 w \left\{ \frac{1}{2(1-\nu)} \left[\frac{(z-d)^2}{R_1^4} + \frac{z^2 - 2dz - d^2}{R_2^4} + \frac{8dz(d+z)^2}{R_2^6} \right] + \frac{1-2}{4(1-\nu)} \left(\frac{1}{R_2^2} - \frac{1}{R_2^2} + \frac{4z(d+z)}{R_2^4} \right) \right\} d\alpha, \tag{57}$$

where R_1, R_2, d and C_2 are given by equations (52)–(54). The upper and lower limits A_L and B_L are given by the following equation:

$$A_L = -B \quad \text{and} \quad B_L = B. \tag{58}$$

The s-sin model strip foundation [see Fig. 5(a)] has a curvature which is a function of sin s as

$$z = \sin^2(\pi s). \tag{59}$$

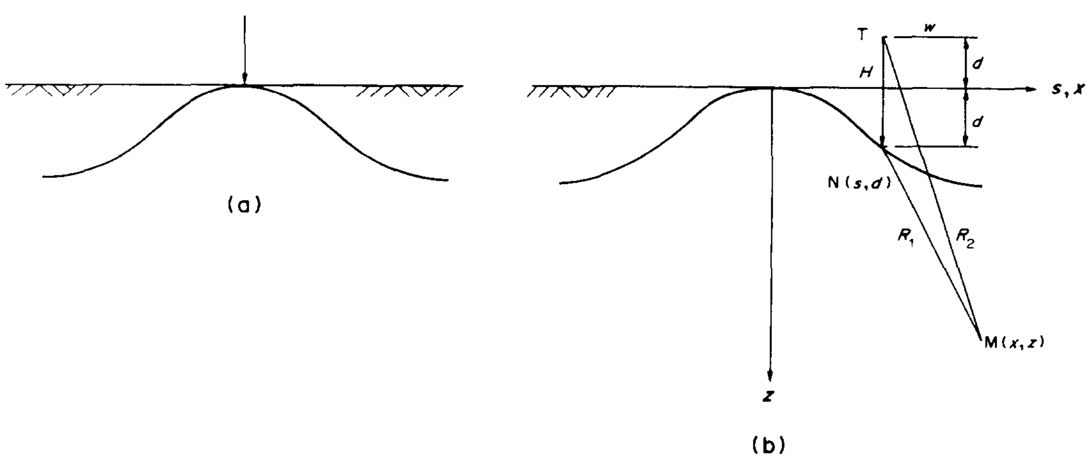

Fig. 5.(a) s-sin strip footing. (b) Vectors and stresses.

The vector analysis for this model [see Fig. 5(b)], is similar to the analysis made for the triangular models. Again, a rigid foundation is considered and the load intensity is linearly proportional to the unit length ds.

Referring to equations (55)–(57), the moduli of the vectors $\overline{\mathrm{MN}}$ and $\overline{\mathrm{MT}}$ are given by

$$\overline{\mathrm{MN}} = R_1 = [w^2 + (z - d)^2] \tag{60}$$

and

$$\overline{\mathrm{MT}} = R_2 = [w^2 + (z + d)^2], \tag{61}$$

where

$$d = \sin^2(\pi s) \tag{62}$$

and

$$w = s - x, \tag{63}$$

s varies between $-B$ and B.

the stresses σ_z, σ_x and τ_{xz} were computed by the program at point M by substituting equations (59)–(63) in equations (55)–(57), respectively.

RESULTS

Five main programs were coded to perform the above-mentioned analysis and to generate results for the purpose of comparison. Each program has 9 subroutines to compute the geometric characteristics of each foundation model. The computer results were tested first with experimental results deduced from a small model, where good agreement with theoretical models was achieved. The computer programs were then used to generate data for the purpose of comparison between the foundation models.

Figures 6–9 show typical results of these analyses. From these figures, it can be seen that the cylindrical, triangular and s-sin footings show some improvements compared to the flat one. Such improvements are demonstrated by the fact that these footings (cylindrical, triangular and s-sin) impose fewer stresses (σ_z, σ_x and τ_{xz}) in the soil and also provide better uniformity of the stress distribution.

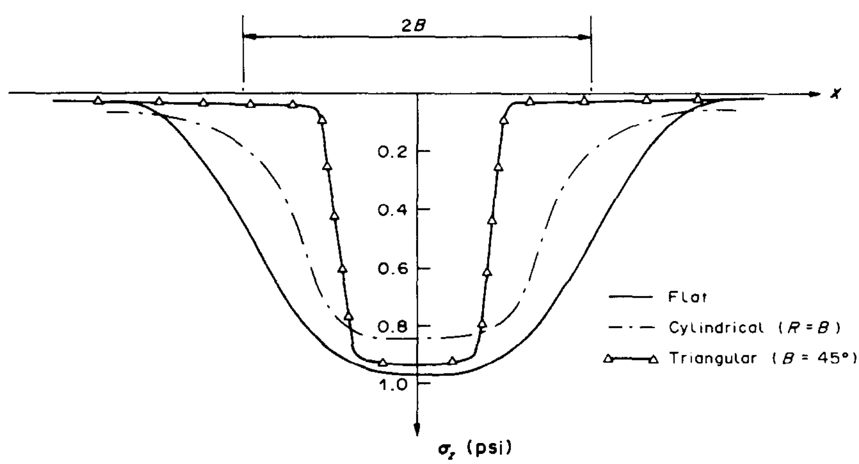

Fig. 6. Vertical stresses σ_z at depth $z = 0.5B$.

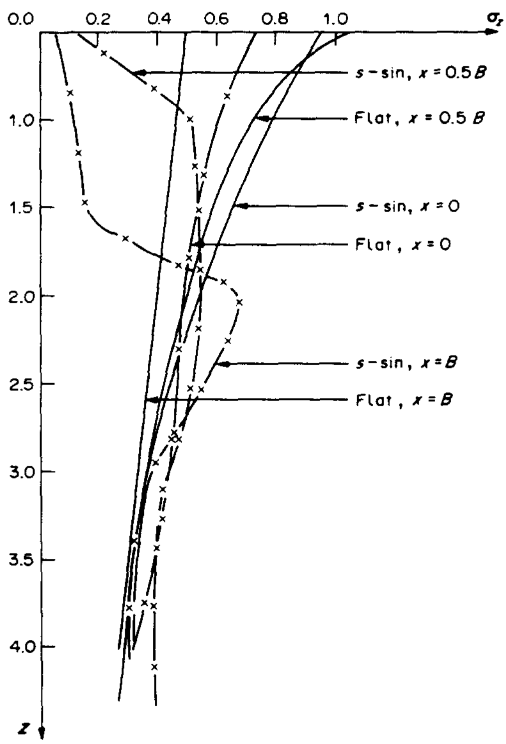

Fig. 7. Variation of vertical stresses σ_z with depth z for flat and s-sin footings.

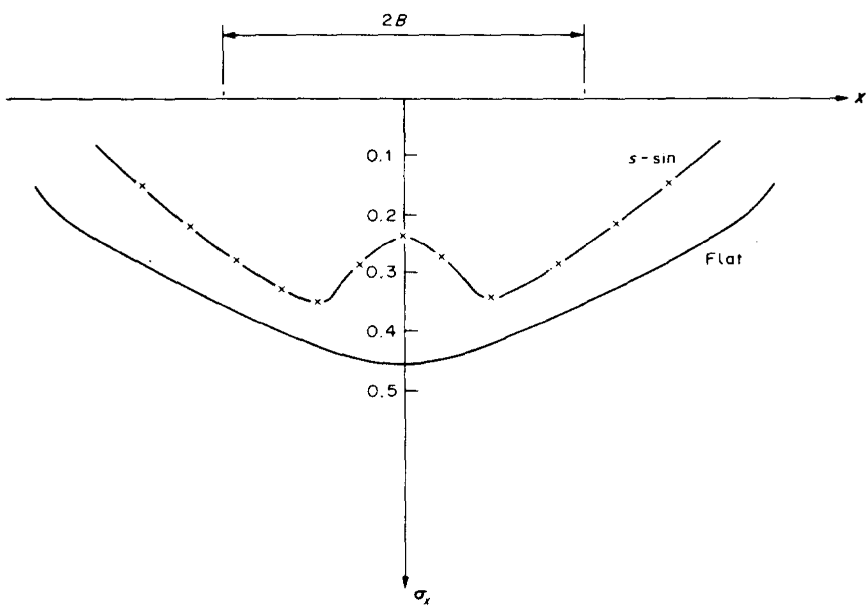

Fig. 8. Horizontal stresses σ_x at $z = 0.5B$.

CONCLUSION

The mathematical models developed in this paper are theoretically sound and when compared experimentally good agreement has been achieved. Further, the proposed foundation models open up a new area in the field of foundation engineering. Based on the fact that under the proposed footing, the soil will experience fewer stresses and even uniform ones compared to the conventional flat footing, it is expected that these footings will have a higher bearing capacity and exercise less settlement so that an economical design can be achieved.

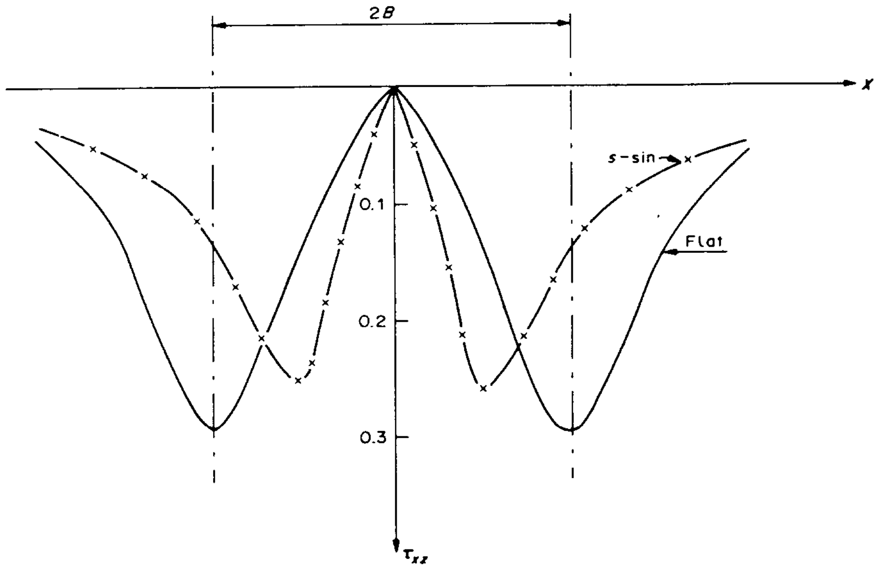

Fig. 9. Shear stresses τ_{xz} at depth $z = 0.5B$.

This study was limited to the above-mentioned shapes, and it is hoped that other shapes will be considered in the future in order to optimize the solution.

Acknowledgement—Financial support from the Natural Science and Engineering Research Council of Canada is gratefully acknowledged.

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